SOLUTIONS TO SAMPLE QUESTIONS

Sample 1-Point Problems

(1) Recognize this as an arithmetic series and try to apply the formula

$$\text{sum} = \frac{(\text{first term} + \text{last term}) \times \#\text{terms}}{2}.$$

The first and last terms are given; to find the number of terms, either check the notation and see that there are 50 terms, or use the formula

$$\#\text{terms} = \frac{\text{last term} - \text{first term}}{\text{common difference}} + 1 = \frac{99-1}{2} + 1 = 50.$$

Finally,

$$1 + 3 + 5 + 7 + \cdots + 99 = \frac{(1+99) \times 50}{2} = 2500.$$

Alternative Solution. Recognize the pattern:

$$\begin{aligned} 1 = 1 &= 1^2, \\ 1 + 3 = 4 &= 2^2, \\ 1 + 3 + 5 = 9 &= 3^2, \\ 1 + 3 + 5 + 7 = 16 &= 4^2, \\ &\cdots \\ 1 + 3 + 5 + 7 + \cdots (2n-1) &= n^2. \end{aligned}$$

Therefore

$$1 + 3 + 5 + 7 + \cdots + 99 = 50^2 = 2500.$$

(2) There are many ways to get the answer. A common approach can be to recognize 1) the spacing between each two consecutive hour marks is $\frac{360^\circ}{12} = 30^\circ$, and 2) the hour hand at 3:40 pm is $\frac{60-40}{60} = \frac{1}{3}$ hours away from number 4. Therefore, the obtuse angle between the minute hand and the hour hand is

$$30^\circ \times (4 + \frac{1}{3}) = 130^\circ.$$

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(3) Notice that the difference between the two ages stays constant at 8 years. Therefore, Anfisa's age is double her sister's when her sister is 8 years old, which is 6 years from now.

Sample 2-Point Problems

(1) The circle's area is

$$A = \pi r^2 = \pi \cdot 2^2 = 4\pi.$$

The side length of the equilateral triangle is $2\sqrt{3}$ (this can be done by drawing a perpendicular from O to one of the vertices and recognize the $30 - 60 - 90$ right triangle, as in Figure 1). It is then easy to calculate the area of the triangle, either using the formula

$$A = \frac{\sqrt{3}}{4}a^2 = 3\sqrt{3}$$

or finding the height on any of the sides to be 3 and use $A = \frac{1}{2}bh$. Subtracting the area of the triangle from the circle, we get that the area of the shaded region is

$$4\pi - 3\sqrt{3}.$$

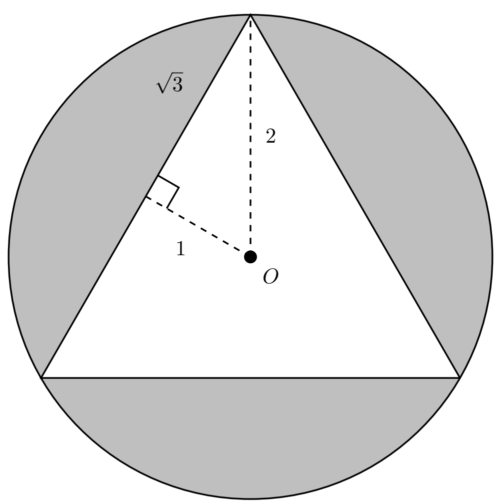

Figure 1

(2) The number of consecutive ending 0's is determined by the number of 2 and 5 factors in 100! (in sophisticated terms, $v_2(100!)$ and $v_5(100!)$. Since it's clear that the number has more factor 2's than factor 5's, it suffices to count the number of factor 5's.

All numbers less than or equal to 100 that's divisible by 5 contributes a factor 5 to 100!; there are 20 of them. Numbers divisible by 25 contribute an extra factor of 5; there are 4 of them. In total, 100! has

$$\frac{100}{5} + \frac{100}{5^2} = 20 + 4 = 24$$

factor 5's and therefore 24 consecutive ending 0's.

(3) We can assume the speed of Anfisa during the 4-hour drive to be x mph. We set up and solve the equation

$$5x = (5 - 1)(x + 15),$$
$$x = 60.$$

Therefore the distance between City A and City B is

$$5 \text{ hrs} \times 60 \text{ mph} = 300 \text{ miles}.$$

Alternative Solution Without Using Equations. Notice that the time to travel a fixed distance is inversely proportional to the speed. The time ratio between the two trips is 5 : 4, so the speed ratio should be 4 : 5. Therefore the difference, 15 mph, correspond to 1 part (if the initial speed is 4 parts and the returning speed is 5 parts). The original speed is $4 \times 15 = 60$ mph and we can calculate the distance to be 300 miles.

Sample 3-Point Problems

(1) Let R be the radius of the larger circle and r be the radius of the smaller circle. Connect O to the point of tangency and one endpoint of the chord, as in Figure 2. It is intuitive (and can be proven with the Perpendicular Chord Bisector Theorem) that the point of tangency is the midpoint of the chord and that the chord is perpendicular to the inner radius. Therefore by the Pythagorean Theorem,

$$R^2 - r^2 = \left(\frac{24}{2}\right) = 144,$$

and the area of the shaded area (an annulus) is simply the difference between the two circles:

$$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = 144\pi.$$

Notice that we don't actually need the values of R and r.

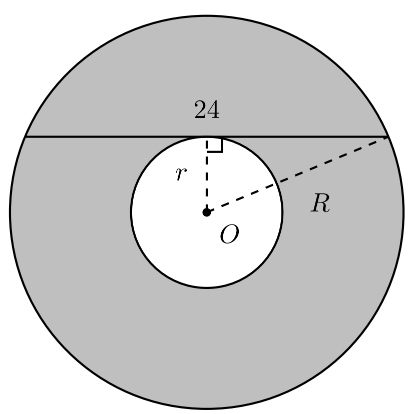

Figure 2

(2) Use difference of squares on each of the factors, and we have

$$
\begin{aligned}
&\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{100^2}\right)\\
=&\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\\
&\qquad\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\cdots\left(1-\frac{1}{100}\right)\left(1+\frac{1}{100}\right)\\
=&\frac{1}{2}\times\frac{3}{2}\times\frac{2}{3}\times\frac{4}{3}\times\frac{3}{4}\times\frac{5}{4}\times\cdots\times\frac{99}{100}\times\frac{101}{100}\\
=&\frac{1}{2}\times\frac{101}{100}\\
=&\frac{101}{200}.
\end{aligned}
$$

(3) First, determine the radius of the circle that circumscribes the triangle to be $\frac{3\sqrt{5}}{\sqrt{3}}=\sqrt{15}$.

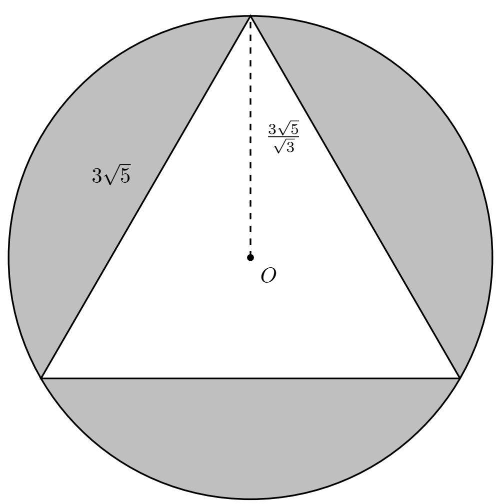

FIGURE 3

Now envision that you're looking at the sphere horizontally so that the plane containing the sphere becomes one base of the shown right triangle. The distance of the plane to the origin is the other base, and the hypotenuse is the radius *of the sphere*. Therefore, the distance is

$$\sqrt{6^2 - (\sqrt{15})^2} = \sqrt{21}.$$

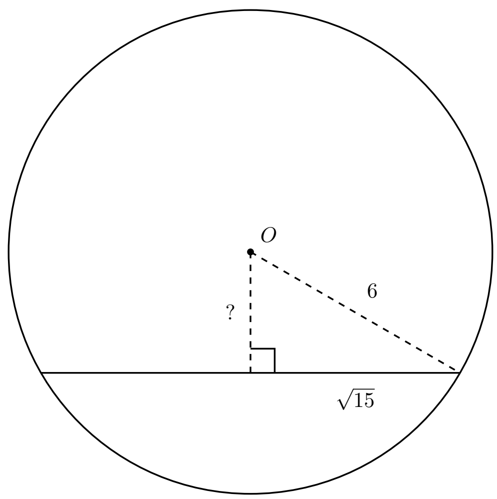

FIGURE 4